Department of Mathematics and Computer Science
Washington College
Fall 2009 – Spring 2010

MATH SENIOR CAPSTONE PROBLEMS AND JOURNAL ARTICLES

7 Problems Leftover from Previous Years

F0910-01 Prove that triangle ABC is equilateral if and only if $\tan A + \tan B + \tan C = 3\sqrt{3}$.

F0910-02 Find two functions of x that are differentiable everywhere and which have the property that the derivative of their quotient is the quotient of their derivatives.

F0910-03 Let x, y, and z be positive real numbers with $x + y + z = xyz$. Find the minimum value of $\sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{1+z^2}$ and find all (x, y, z) for which the minimum occurs.

F0910-04 Let P be a point inside equilateral triangle T. Show that the sum of the three distances from P to the sides of T is constant.

F0910-05 Let $p = \log_8 3$ and $q = \log_3 5$. Find $\log_{10} 5$ in terms of p and q.

F0910-06 Use the power series for $\arctan x$ to prove the following expression for π as the sum of an infinite series.

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

F0910-07 Let ΔABC be a right triangle with a right angle at C. Let D and E be points on the hypotenuse AB such that $|BD| = |DE| = |EA|$ and for some value of x with $0 < x < \frac{\pi}{2}$ suppose that $|DC| = \sin x$ and $|EC| = \cos x$. Find $|AB|$.

New Problems for Fall 2009

F0910-08 **Given:** x, y, and z are primitive Pythagorean triples

Prove: xy is divisible by 12

F0910-09 **Given:** $n \in \mathbb{Z}^+$

Prove: If n is a perfect square, then n is not a perfect number.

F0910-10 Given

$$A = \begin{bmatrix} a & b & c \\ p & 0 & 0 \\ 0 & q & 0 \end{bmatrix}$$

where b, c, p, q are non-negative real numbers and $a \in \mathbb{R}^+$.

Show that A has only one positive eigenvalue. then find the form of the eigenvectors corresponding to that eigenvalue.

F0910-11 Jane knows that the heights of males at the college she attends are normally distributed with a mean of 70 inches and a standard deviation of 4 inches. The heights of female undergraduates are also normally distributed with a mean of 67 inches and a standard deviation of 3 inches. Male students outnumer female students 3 to 1. Jane has been randomly assigned a partner for a campuswide sporting event. Jane, a basketball player, is 72 inches tall. What is the probability that her partner is taller than she?

F0910-12 **Given:** z is a complex number, $|z| \leq 1$

Prove: $|z-1| + |z+1| \leq 2\sqrt{2}$

F0910-13 Find an equation for the plane tangent to the cone $z^2 = x^2 + y^2$ at the point (a, b, c). Then verify that this tangent plane passes through the origin.

F0910-14 Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3\ 2^{n+2}}.$$

F0910-15 A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with a speed of 115 ft/s at an angle of 50° above the horizontal. Is it a home run? Explain why or why not?

F0910-16 If A is normal matrix and U is unitary matrix, show that $T = U^*AU$ is also a normal matrix.

F0910-17 When its engines are on, the position function for a spaceship is

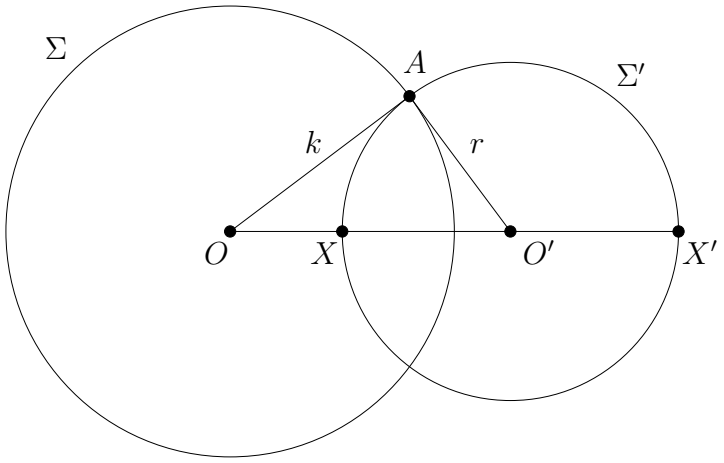
$$\mathbf{r}(t) = (3+t)\mathbf{i} + (2+\ln t)\mathbf{j} + \left(7 - \frac{4}{t^2+1}\right)\mathbf{k}.$$

If the captain wants to coast to a space station that is located a coordinates $(6, 4, 9)$, when should the engines be turned off? Explain.

F0910-18 Consider the closed system of population migration between four cities: Baltimore (B), New York (N), Philadelphia (P), and Washington (W). Suppose that each year 5% of B's population moves to N, 3% of B's population moves to P, 7% of B's population moves to W, 1% of N's population moves to B, 3% of N's population moves to P, 2% of N's population moves to W, 2% of P's population moves to B, 4% of P's population moves to N, 3% of P's poplulation moves to W, 5% of W's population moves to B, 4% of W's population moves to N, and 1% of W's poplulation moves to P.

1. Suppose a person currently lives in Philadelphia. What is the probability that he/she will live in New York after 10 years? Explain.
2. Suppose a person currently lives in Baltimore. What is the probability that he/she will live in Baltimore after 30 years? Explain.

F0910-19 Given circle Σ centered at O with radius k and circle Σ' centered at O' with radius r as shown in the figure below, suppose that X and X' lie on Σ' and also on the line connecting O and O' such that $|OX| \cdot |OX'| = k^2$. Suppose that point A lies on both circles. Prove that $\overline{OA} \perp \overline{O'A}$.



F0910-20 An old riddle runs as follows. An explorer walks one mile due south; turns and walks one mile due east, turns again and walks one mile due north. He finds himself back where he started. He shoots a bear. What color is the bear? The time-honored answer is: "White," because the explorer must have started at the North Pole. But not long ago someone made the discovery that the North Pole is not the only starting point that satisfies the given conditions! Can you think of any other spot on the globe from which one could walk a mile south, a mile east, a mile north and find himself back at his original location?

F0910-21 The props for this problem are a chessboard and 32 dominoes. Each domino is of such size that it exactly covers two adjacent squares on the board. The 32 dominoes therefore can cover all 64 of the chessboard squares. But now suppose we cut off two squares at diagonally opposite corners of the board and discard one of the dominoes. Is it possible to place the 31 dominoes on the board so that all the remaining 62 squares are covered? If so, show how it can be done. If not, prove it impossible.

F0910-22 Imagine that you have three boxes, one containing two black marbles, one containing two white marbles, and the third, one black marble and one white marble. The boxes were labeled for their contents BB, WW, and BW but someone has switched the labels so that every box is now incorrectly labeled. You are allowed to take one marble at a time out of any box, without looking inside, and by this process of sampling you are to determine the contents of all three boxes. What is the smallest number of drawings needed to do this?

F0910-23 A commuter is in the habit of arriving at his suburban station each evening exactly at five o'clock. His wife always meets the train and drives him home. One day he takes an earlier train, arriving at the station at four. The weather is pleasant, so instead of telephoning home he starts walking along the route always taken by his wife. They meet somewhere on the way. He gets into the car and they drive home, arriving at their house ten minutes earlier than usual. Assuming that the wife always drives at a constant speed and that on this occasion she left just in time to meet the five o'clock train, can you determine how long the husband walked before he was picked up?

F0910-24 You have 10 stacks of coins, each consisting of 10 half-dollars. One entire stack is counterfeit, but you do not know which one. (The other stacks contain no counterfeit coins.) You do know the weight of a genuine half-dollar and you are also told that each counterfeit coin weighs one gram more than it should. You may weigh the coins on a scale. What is the smallest number of weighings necessary to determine which stack is counterfeit?

Journal Articles that May Be Presented for Problem Credit

1. Schilling, Mark F. "Do You Know Your Relative Driving Speed?", *Mathematics Magazine*, Vol. 79, No. 2 (Apr. 2006), pp. 131-135. (1 problem credit)

2. Lazer, Alan C. "From the Cauchy-Riemann Equations to the Fundamental Theorem of Algebra," *Mathematics Magazine*, Vol. 79, No. 3 (Jun. 2006), pp. 210-213. (1 problem credit)

3. Brookfield, Gary. "Factoring Quartic Polynomials: A Lost Art," *Mathematics Magazine*, Vol. 80, No. 1 (Feb. 2007), pp. 67-69. (1 problem credit)

4. Nelsen, Roger B. "The Relationship between Hyperbolic and Exponential Functions," *The College Math. Journal*, Vol. 19, No. 1 (Jan. 1988), pp. 54-56. (1 problem credit)

5. Tan, Lin. "The Group of Rational Points on the Unit Circle," *Mathematics Magazine*, Vol. 69, No. 3 (Jun. 1996), pp. 163-171. (1 or 2 problem credits)